Correction(pp.142-144) to

"Proof of $\sin x < x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}, x \in (0, \pi/2)$**"**

Must be:

- **Proof of** $\sin x < x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}, x \in (0, \pi/2)$

Let $x \in (0, \pi/2)$ .Since $\sin x < x$ and $\cos x < 1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4$ then

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} < 2\cdot\frac{x}{2}\left(1 - \frac{(x/2)^2}{2} + \frac{(x/2)^4}{24}\right) = x - \frac{x^3}{8} + \frac{x^5}{384}\ .$$

So, we have

$$\sin x < x - \frac{x^3}{8} + \frac{x^5}{384},\quad x \in (0, \pi/2)\,. \qquad \textbf{(6)}$$

Suppose now that for some positive $a < \dfrac{1}{3!} = \dfrac{1}{6}$ and positive $b < \dfrac{1}{5!} = \dfrac{1}{120}$ the inequality $\sin x \le x - ax^3 + bx^5$ holds for every $x \in (0, \pi/2)$.

Then

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} < 2\left(\frac{x}{2} - a\left(\frac{x}{2}\right)^3 + b\left(\frac{x}{2}\right)^5\right)\left(1 - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1}{24}\left(\frac{x}{2}\right)^4\right) =$$

$$x - \left(\frac{a}{4} + \frac{1}{8}\right)x^3 + \left(\frac{a}{32} + \frac{b}{16} + \frac{1}{384}\right)x^5 - \left(\frac{x}{2}\right)^7\left(\frac{1}{12}a + b\left(1 - \frac{x^2}{48}\right)\right) <$$

$$x - \left(\frac{a}{4} + \frac{1}{8}\right)x^3 + \left(\frac{a}{32} + \frac{b}{16} + \frac{1}{384}\right)x^5$$

because for for any $x \in (0, \pi/2)$ holds inequality $\dfrac{1}{12}a + b\left(1 - \dfrac{x^2}{48}\right) > 0$.

Thus, the non-strict inequality $\sin x \le x - ax^3 + bx^5$ (which we know to be true when $a = 1/8$ and $b = 1/384$) yields the strict inequality

$$\sin x < x - \left(\frac{a}{4} + \frac{1}{8}\right)x^3 + \left(\frac{a}{32} + \frac{b}{16} + \frac{1}{384}\right)x^5 \qquad \textbf{(7)}$$

If we write $a = \dfrac{1}{6} - p, b = \dfrac{1}{120} - q$ and denote $\sin x - x + \dfrac{x^3}{6} - \dfrac{x^5}{120}$ via $r(x)$ then

$$\sin x \le x - ax^3 + bx^5 \iff \sin x \le x - \left(\frac{1}{6} - p\right)x^3 + \left(\frac{1}{120} - q\right)x^5 \iff$$

$$\sin x - x + \frac{x^3}{6} - \frac{x^5}{120} \le px^3 - qx^5 \iff r(x) \le px^3 - qx^5,\ \text{where } p, q > 0$$

and since

$$\frac{a}{4} + \frac{1}{8} = \frac{1}{6} - \frac{p}{4},\ \frac{a}{32} + \frac{b}{16} + \frac{1}{384} =$$

$$\frac{1}{32}\left(\frac{1}{6} - p\right) + \frac{1}{16}\left(\frac{1}{120} - q\right) + \frac{1}{384} = \frac{1}{120} - \frac{p}{32} - \frac{q}{16}$$

we obtain **(7)** $\iff \sin x < x - \left(\dfrac{1}{6} - \dfrac{p}{4}\right)x^3 + \left(\dfrac{1}{120} - \dfrac{p}{32} - \dfrac{q}{16}\right)x^5 \iff$

$r(x) < \frac{p}{4}x^3 - \left(\frac{p}{32} + \frac{q}{16}\right)x^5.$

So, we have shown that the inequality $r(x) < px^3 - qx^5$ and even $r(x) \leq px^3 - qx^5$ implies the inequality

$$r(x) < \frac{p}{4}x^3 - \left(\frac{p}{32} + \frac{q}{16}\right)x^5.$$

Due to inequality **(6)** with $a = \frac{1}{8}$ and $b = \frac{1}{384}$ the initial value of p is $\frac{1}{6} - \frac{1}{8} = \frac{1}{24}$ and the initial value of q is $\frac{1}{120} - \frac{1}{384} = \frac{11}{1920}$.

Let sequences $(p_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ be as follows

$$p_{n+1} = \frac{p_n}{4},\ q_{n+1} = \frac{p_n}{32} + \frac{q_n}{16}, n \in \mathbb{N}\ , p_1 = \frac{1}{24}, q_1 = \frac{11}{1920}.$$

Thus, we can see that for any $x \in (0, \pi/2)$ holds inequality $r(x) < p_n x^3 - q_n x^5 < p_n x^3$ (because $q_n x^5 > 0$ for any $n \in \mathbb{N}$)

Noting that $p_n = \frac{1}{24} \cdot \frac{1}{4^{n-1}} = \frac{1}{3 \cdot 2^{2n+1}}, n \in \mathbb{N}$ we obtain inequality

$$r(x) < p_n x^3 = \frac{x^3}{3 \cdot 2^{2n+1}}, n \in \mathbb{N}.$$

Since $\frac{1}{3 \cdot 2^{2n+1}}$ can be arbitrary small with increasing n then $\frac{x^3}{3 \cdot 2^{2n+1}} < \frac{(\pi/2)^3}{6 \cdot 4^n}$ can be arbitrary small with increasing n as well then, applying **Proposition** to $f(x) = r(x)$ we obtain inequality $r(x) \leq 0, x \in (0, \pi/2)$ which equivalent to inequality $\sin x \leq x - \frac{1}{6}x^3 + \frac{1}{120}x^5, x \in (0, \pi/2)$ and, since $\sin x \leq x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ yields

$$\sin x < x - \left(\frac{1}{6} \cdot \frac{1}{4} + \frac{1}{8}\right)x^3 + \left(\frac{1}{6} \cdot \frac{1}{32} + \frac{1}{120} \cdot \frac{1}{16} + \frac{1}{384}\right)x^5 =$$

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

we finally get strict inequality $\sin x < x - \frac{1}{6}x^3 + \frac{1}{120}x^5.$